Continuous Dirichlet Solutions and Infinite Graphs

Jack Coughlin

August 11, 2011

1 Introduction

In complex analysis, the Poisson Integral formula provides a convenient way to extend harmonic functions from the boundary of a domain to the interior. On the unit disk, we have

$$\tilde{h}(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \varphi)h(e^{i\varphi})\, d\varphi$$

As it happens, $\tilde{h}$ is also harmonic, and, for piecewise continuous boundary values h, the radial limit of $\tilde{h}$ is the average of the right and left hand limits of h. The Poisson kernel $P_r(\theta)$ is a family of functions indexed by radius, which are domain specific. On the unit disk, we have

$$P_r(\theta) = \frac{1 - |z|^2}{|1 - z|^2}.$$

It would be handy if such a formula existed for γ-harmonic functions on graphs. Even the notion of radial limit is difficult to generalize to all graphs, so we will restrict our attention in this paper to a certain subset of graphs which behave nicely. As a preview, our goal is a solution to the Dirichlet problem, and an analogue to the Poisson Integral formula, on infinite trees.

2 Preliminaries

2.1 Graphs and Networks

Definition 2.1. An infinite graph, although throughout this paper we will simply refer to graphs for simplicity's sake, is a double $G = (V, E)$, where V is an infinite set of vertices, and $E \subset V \times V$ is a collection of pairs of vertices. One vertex $v_0 \in V$ is designated as the origin.
Definition 2.2. A network $\Gamma = (G, \gamma)$ is a graph G coupled with a function $\gamma : E \to \mathbb{R}^+$ representing the conductivities of connections in an electrical network. We require that γ be bounded above and below, thus, $0 < c \leq \gamma \leq C < \infty$.

2.2 Γ-Harmonic Functions

Definition 2.3. A potential function, or voltage function, is simply a function $u : V \to \mathbb{R}$. This is intended to model voltages on an electrical network, with the weightings representing conductivities.

Definition 2.4. A voltage function u is called γ-harmonic if, for each interior vertex v, we have

$$\sum_{(v,v') \in G_{int} \times G} \gamma(vv')(u(v) - u(v')) = 0.$$

The sum over the Cartesian product makes sense if we simply define $\gamma = 0$ on pairs of vertices that do not form edges. Such a function satisfies the physical law that current should neither be created nor destroyed by the system. Current can enter and exit via boundary nodes, but not interior nodes. Harmonic functions satisfy two useful simple properties:

Lemma 2.5 (Mean Value Property)**.** *If v is an interior vertex of degree k, and u is a γ-harmonic function, then*

$$u(v) = \frac{1}{\sum_{v' \sim v} \gamma(vv')} \sum_{v' \sim v} \gamma(vv')u(v'),$$

where $v' \sim v$ means that v' is a neighbor of v. That is, the value of a harmonic function at a point is a weighted average of its values at neighboring points.

Proof. The proof is a simple rearrangement of the expression for the definition of γ-harmonicity. □

Corollary 2.6. No interior point of a γ-harmonic function can be a local extremum of the function.

Proof. If v is a local maximum (resp. minimum), then $u(v)$ is greater (resp. lower) than each $u(v')$, violating the mean value property.

Lemma 2.7 (Maximum Principle)**.** *If Γ is a graph with boundary (not necessarily finite), and u is a γ-harmonic function whose boundary values lie in the interval $[a, b]$ and whose limits along infinite paths lie in the interval $[c, d]$, then u takes values exclusively in the interval $[\min\{a, c\}, \max\{b, d\}]$. That is, u takes its maximum and minimum on the boundary, or at infinity.*

Proof. Suppose without loss of generality that $u(v_0) < a \leq c$. Either v_0 is a local minimum, such that all its neighbors v' verify $u(v_0) \leq u(v')$, or one of its neighbors takes a lower value. We can continue this argument to produce a path of decreasing values. This path must terminate in the interior. It cannot end at the boundary, by assumption on the values at the boundary. And it cannot have infinite length, again by assumption. So it terminates at a local minimum.

But γ-harmonic functions do not have local extrema in the interior, so this is a contradiction. The argument against a maximum is exactly symmetric. □

3 Paths and Boundaries

Definition 3.1. A path p in G is a sequence of vertices $\{v_n\}$, beginning at the origin of G such that each pair of adjacent vertices $v_0v_1, v_1v_2 \cdots \in E$. The path may be self-intersecting, in the sense that we do not require that $\forall i \neq j, v_i \neq v_j$. These paths, then, are essentially random walks on the graph.

Definition 3.2. We say that a path p **escapes** if, for each $M \in \mathbb{N}$, the number of edges in p at distance of at most M from v_0, counting multiplicity, is finite.

Lemma 3.3. *If $\hat{p}$ has no recurring edges, it escapes.*

Proof. The proof of the latter claim is easy. Since each vertex has finite degree, the number of edges within M of v_0 is finite, and since they cannot be reused, $\hat{p}$ eventually leaves the region. □

We want a way to categorize the infinite paths, in terms of how they behave at infinity. Thus, we define a relation between two paths, which we will prove is an equivalence under certain circumstances.

Definition 3.4. Two paths p and q are related by $\sim$, and we write $p \sim q$, if p and q share infinitely many edges.

Lemma 3.5. *If G is a tree, or a graph with no cycles, and $\mathbb{E}$ denotes the set of all escaping paths, then $\sim$ is an equivalence relation on $\mathbb{E}$.*

Proof. Let $p, q \in \mathbb{E}$. Clearly $p \sim p$, and $p \sim q \iff q \sim p$. Thus, we only need show that $\sim$ is transitive. Suppose $s \in \mathbb{E}$ satisfies $p \sim s$, and $s \sim q$. We need to show $p \sim q$.

Observe that if a path b shares two arbitrarily distant edges e_1 and e_2 with another path a, then a and b must coincide at all intermediate edges. If they do not, and there is more than one route from e_1 to e_2, then concatenating the two routes produces a cycle, which is impossible.

Now, given that s is an escaping infinite path, note that all infinite paths covered by s are similar to each other. Suppose there are two such paths a and b with only finitely many edges in common. Then, by the acyclicity of G, a and b contain edges at arbitrary distance from each other (if both edges are D from the origin, then they are $2D$ from each other). But because s is connected, and begins at the origin and not at infinity, there must be infinitely many traversals through the origin connecting edges of a and b at successively greater distances.

Now, by assumption, $p \sim s$ and $q \sim s$, so both p and q cover infinite subpaths which are also covered by s. Thus, both p and q contain edges arbitrarily far along the same infinite subpath of s. Pick two edges p_1 and p_2 contained in p at very great distance from each other, far enough apart that we can find two edges in q arbitrarily far apart from each other, both within the unique route connecting p_1 and p_2. Then p and q coincide at arbitrarily many edges along that unique route, as required. □

Henceforth, we work exclusively on trees. Now, the relation $\sim$ effects a partition of $\mathbb{E}$ into equivalence classes. By the following lemma, we can represent each class by a unique special element.

Lemma 3.6. *Each equivalence class* $[p]$ *of* $\mathbb{E}$ *generated by* $\sim$ *contains exactly one path* $\hat{p}$ *with no recurrent edges.*

Proof. Each element of $[p]$ contains edges arbitrarily far from the origin. For the edges sufficiently distant, there is a unique, direct path with no recurrent edges connecting them to the origin. Take the countable union of all these unique paths; it is unique, by the acyclicity of the tree. Designate this unique non-recurrent path by $\hat{p}$. □

We denote the set of all representative paths, corresponding exactly to the set of equivalence classes, by $\hat{\mathbb{E}}$. Because all the paths are escaping to infinity, and they are all effectively distinct at infinity, it makes sense to consider this set of paths to be the boundary of the graph. Thus, we will often write ∂G or $\partial\Gamma$ to mean $\hat{\mathbb{E}}$. If we are going to use an integral formula of some kind to integrate along the boundary of the graph, we need an ordering of the paths in the boundary for that to make any sense.

Lemma 3.7. *Let* G *be a planar infinite tree, with origin* v_0*. Then there is a well-ordering of the set of non-recurrent infinite paths* $\hat{\mathbb{E}}$*.*

Proof. Beginning at the origin, select one edge incident to v_0 to be the left-most edge. Proceed along this edge, and at each vertex, turn left. The path produced in this way will be the infimum of $\mathbb{E}$ with respect to our ordering. We write $p \leq q$ if, at the vertex where p and q diverge, q follows an edge to the right of p. If p and q never diverge, then $p = q$. Given any nonempty subset of $\mathbb{E}$, there is a leftmost element, produced by simply following, at each vertex, the leftmost edge contained in a path of the subset. Thus, $\leq$ defined in this way is a well-ordering. □

We can also associate to the boundary a pseudometric.

Definition 3.8. Let p and q be two finite paths from the origin, both of length i. The well-ordering of infinite paths from the origin also provides an ordering of finite paths, so denote by $p - q$ the number of paths s satisfying $p < s \leq q$. We define the **standard metric** on paths in $\partial\Gamma$ to be

$$d(\hat{p}, \hat{q}) = \|p - q\| = \lim_{i\to\infty} (p_i - q_i)/|T_i|,$$

where $|T_i|$ is the number of subpaths of length i, or, equivalently, the number of vertices in Γ at distance i from the origin. If $\|p-q\| > 0$ and $p \leq q$, then we write $p < q$.

Definition 3.9. A continuous path numbering $\zeta : \partial\Gamma \to \mathbb{R}$ is function which is both strictly increasing with respect to the standard ordering on $\partial\Gamma$, and continuous with respect to the standard metric. By strictly increasing, we mean that $p < q \implies \zeta(p) < \zeta(q)$.

Lemma 3.10. *The image of a continuous path numbering f is an interval.*

Proof. The proof is nearly identical to the Intermediate Value Theorem. Let $f(a)$ and $f(b)$ be the least and greatest values of f. Let $f(a) \leq u \leq f(b)$. We prove that there is a value c such that $f(c) = u$. Indeed, let $c \in S = \sup\{x|f(a) \leq f(x) \leq u\}$. Clearly $c \in \partial\Gamma$, because $\partial\Gamma$ is complete. Suppose for contradiction that $f(c) - u > 0$. Then since f is continuous, there is $\delta > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| > f(c) - u$. That is, $f(x) > u$ for $x \in (c - \delta, c + \delta$. Thus, $c - \delta$ is an upper bound for S. On the other hand, if we suppose that $f(c) < u$, then $u - f(c) > 0$, and we can again find x such that $f(x) < u$, but $x \in (c - \delta, c + \delta)$. This is again a contradiction.

Corollary 3.11. It follows, from the fact that translations and dilations are continuous, that if there exists a continuous path numbering from $\partial\Gamma$ to some non-degenerate interval, there exists a continuous path numbering from $\partial\Gamma$ to any non-degenerate interval.

4 Measures on $\partial\Gamma$

In this section, I develop two different measures on the boundary of Γ. The first is very simple, deriving directly from the continuous path numberings discussed above.

Definition 4.1. The uniform measure μ on $\partial\Gamma$ with respect to a continuous path numbering $\zeta$is simply the Lebesgue measure defined with respect to the image of ζ. Thus, if $E \subset \partial\Gamma$, $\mu(E) = \lambda(\zeta(E))$.

The second measure is a probability measure on infinite paths, for which we must first define a σ-algebra.

Proposition 4.2. The set of all random walks beginning at an arbitrary vertex v, denoted X, admits an algebra $\mathfrak{E}_v$ consisting of the sets E of random walks which begin with some fixed finite sequence of moves. That is, to each finite sequence S of moves beginning at v corresponds a set E_S of random walks which begin with that sequence, and the collection of all such sets generates an algebra closed under finite unions and complements.

Proposition 4.3. The **probability measure** ν_v is a finite premeasure on $\mathfrak{E}_v$. Define E_S as above, then the probability measure of E_S is

$$\nu_v(E_S) = \prod_{v_i \in S} \frac{\gamma(v_i v_{i+1})}{\sum_{v' \sim v_i} \gamma(v' v_i)}.$$

That is, at each vertex, multiply by the weighted probability of following the next edge in S. In other words, $\nu_v(E_S)$ is the probability that a random walk beginning at the origin will follow the path S. We define ν on the sets which are generated by finite unions of sets like E_S by simply requiring that it be finitely additive over disjoint unions. This is well-defined, because, for any two generating sets E_1, E_2, either $E_1 \subset E_2, E_2 \subset E_1$, or $E_1 \cap E_2 = \varnothing$. That it is indeed a probability measure follows from the observation that

$$\partial\Gamma = \bigcup_{1}^{d(v)} E_{e_i},$$

where $\{e_i\}$ is the set of edges immediately incident to v_0, and

$$\nu_v(X) = \sum_{1}^{d(v)} \nu_v(E_{e_i}) = \sum_{1}^{d(v)} \frac{\gamma(vv_i)}{\sum_1^{d(v)} \gamma(vv_i)} = 1.$$

□

Lemma 4.4. *Denote by $\mathfrak{F}_v$ the σ-algebra generated by $\mathfrak{E}_v$. If $T \subset \partial\Gamma = \hat{\mathbb{E}}$ is a set of escaping paths which is determined by its finite truncations, then $T \in \mathfrak{F}_v$.*

Proof. We will generate T from the basis elements of $\mathfrak{F}_v$. Let T_n be the set of vertices in T at distance n from v, and let x_i be a vertex in that set. The set of all paths from v that meet x_i at their j^{th} step is clearly in $\mathfrak{F}_v$, denote it by $M_i^j(n)$. Now, take the countable union of these sets over j:

$$M_i(n) = \bigcup_{j=1}^{\infty} M_i^j(n) \in \mathfrak{F}_v.$$

Now, take the countable (actually, finite) union of the $M_i(n)$'s over all vertices $x_i \in T_n$:

$$M(n) = \bigcup_{i=1}^{\infty} M_i(n) \in \mathfrak{F}_v.$$

This set M is actually specific to the index n, so we now take the countable intersection over n:

$$M = \bigcap_{n=1}^{\infty} M(n) \in \mathfrak{F}_v.$$

This is the set of all paths which meet a point in every subset T_n at some point in their meanderings. Some of these paths spend arbitrarily long near the origin, so we need to be able to excise them from our set. But we can do this, because the set of paths that do not escape, $\mathbb{E}^C$, is in $\mathfrak{F}$. Indeed, take the countable union (over K) of the countable intersections (over n) of the complements of sets of paths that end up more than K steps away from the origin after n steps. So,

$$T = \bigcap_{n=1}^{\infty} M(n) \setminus \mathbb{E}^C \in \mathfrak{F}_v.$$

□

Lemma 4.5. *The premeasure $\nu(\mathfrak{E}_v)$ as defined above extends uniquely to a measure on $\mathfrak{F}_v$.*

Proof. Because $\mathfrak{F}_v$ is generated by $\mathfrak{E}_v$, we can apply [1] p. 30, Theorem 1.14. Thus, there exists a unique extension of ν to $\mathfrak{F}_v$, which we also call ν.

We can relate the two measures to each other by absolute continuity, defined in [1], p. 83.

Lemma 4.6. *The measure ν is absolutely continuous with respect to μ: $\nu \ll \mu$.*

Proof. Suppose for the first case that $\mu(E) = 0$. Then we can cover E with intervals of arbitrarily small total length. These intervals correspond to subtrees of Γ which "cover" the paths in E, which we can begin arbitrarily far from v_0 and still have cover the tails of $\hat{p} \in E$. Fix $\epsilon > 0$, this fixes a covering of E by intervals and subtrees. Let x be the base of a subtree T_ϵ which covers the tail of some $\hat{p} \in E$. Each infinite walk which is equivalent to a path in T_ϵ meets x a final time, otherwise it does not escape.
If we lower ϵ enough that T_ϵ is now one level below where it was, such that we have eliminated all but one exiting edges from x as a possibility, then not all the paths which met x will survive in the smaller T_ϵ. The proportion of those which do is precisely the ratio of the chosen edge's conductivity to the total conductivities out of x. Because γ is bounded above and below, and Γ is locally finite, this proportion is bounded above by some $q < 1$. Taking ϵ arbitrarily small, we repeat this process infinitely many times. Finally, the total measure of all the paths which survive the process, having been multiplied by q^n at the most, goes to 0.

Corollary 4.7. The Radon Nikodym theorem ([1] p. 84) guarantees the existence of a function f such that

$$\nu(E) = \int_E f(x)\, d\mu.$$

This function is a probability density function, and we write $P_v(x)$.

Lemma 4.8. *The function $P_v(x)$, which is in fact a family of functions indexed by vertices, is γ-harmonic with respect to v except on at most a μ-null set.*

Proof. Let E be a subset of $\partial\Gamma$. Then we have

$$\nu_v(E) = \int_E P_v(x)\,dx = \frac{1}{k}\sum_{v'\sim v} \nu_{v'}(E) = \frac{1}{k}\int_E \sum_{v'\sim v} P_{v'}(x)\,dx$$

If the sum of the density functions of the neighboring vertices differs from $P_v(x)$ on a set of nonzero measure, then we set E equal to that set, and violate the foregoing equality. □

Lemma 4.9. *Let $\hat{p} \in \hat{\mathbb{E}}$, and index the vertices of $\hat{p}$ by $\{v_i\}$. Then*

$$\lim_{i\to\infty} P_{v_i}(\xi(\hat{p})) = \infty.$$

Proof. Take v_i far out in the path. It is the apex of a subtree of the graph, which defines an interval in $\partial\Gamma$. Call this interval I_i. Now form the interval I_j, where $j > i$, and $I_j \subset I_i$. The ν-measure of $[0,1] - I_i$ from v_j is nonzero but small, call it $q > 0$. Incrementing j by 1, we multiply q by the non-unity probability of a random walk starting at v_{j+1} moving to v_j; this probability is bounded by $\alpha < 1$, from the bounds on γ. Thus, as $j \to \infty$, $q \to 0$. And as $i \to \infty$, $I_i \to \xi(\hat{p})$. In other words, ν tends to vanish on all but a single point of $[0,1]$. Because the total measure is 1, however, this means that $P_{v_i} \to \infty$. □

We are now ready, after much trial and travail, to state the main result.

Theorem 4.10. *Let Γ be an expansionary tree with conductivities bounded above and below, and let $\phi : [0,1] \to \mathbb{R}$ be the piecewise continuous function assigning values to the boundary at infinity. Then there exists a unique γ-harmonic voltage function u such that for each path $\hat{p} = \{v_i\} \in \hat{\mathbb{E}}$,*

$$\lim_{i\to\infty} u(v_i) = \phi(\xi(\hat{p}))$$

wherever ϕ is continuous. Furthermore, the explicit expression for u is

$$u(v) = \tilde{h}(v) = \int_0^1 P_v(x)\phi(x)\,d\mu.$$

Proof. That $\tilde{h}(v)$ is γ-harmonic follows immediately from Proposition 3.4. We only need to show the limit. Let x_0 be a point in $[0,1]$ where ϕ is continuous. Because P_v behaves like a Dirac δ-function, for every $\epsilon > 0$ there is extremely large N such that for $n > N$, wherever $P_{v_n}(x_0) > \epsilon$, ϕ is continuous. Call the corresponding interval I_ϵ. The integral of P_{v_n} outside I_ϵ is at most ϵ, so its integral inside I_ϵ is $1-\epsilon$. By the continuity of ϕ on I_ϵ, we can bound its values by $\phi(x_0) - \delta \leq \phi(I_\epsilon) \leq \phi(x_0) + \delta$, where δ goes to 0 with ϵ. Thus,

$$(1-\epsilon)(\phi(x_0) - \delta) \leq \int_{I_\epsilon} P_{v_n}(x)\phi(x)\, d\mu \leq (1-\epsilon)(\phi(x_0)+\delta).$$

As $\epsilon, \delta \to 0$, the integral is sandwiched between the two limits, and approaches $\phi(x_0)$. The error term from outside I_ϵ is vanishing with ϵ, so

$$\lim_{i\to\infty} \int_0^1 P_{v_i}(x)\phi(x)\, d\mu = \phi(\xi(\hat{p})).$$

□

5 Conclusion

Proposition 5.1. $\mu \ll \nu$. If this is true, then subsets of $\partial\Gamma$ with null ν-measure are countable, or negligible with respect to μ. This improves the set of graphs for which the above results hold. Indeed, if it is true, then we need not worry about graphs with rays of width 1, which have ν-measure 0. That it should be true can be seen by observing that it is equivalent to the question of whether uncountable trees are transient. They should be, because the conductivities are bounded above and below, and the number of edges facing downwards in such a tree must dominate the number facing upwards at some point. On average, the probability of going upwards should be less than 1/2, while the probability of moving downward should be more than 1/2.

The results presented above require clarification in terms of the class of graphs which can be considered, requirements on the path numberings ζ, and so forth. The paper should be considered as a blueprint for more rigorous examination of the issues.

References

[1] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, New York, 1984.